Ekstremne vrijednosti f-ja dviju promjenjivih

Neka je data f-ja $z = f(x, y)$.

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial y} &= 0 \end{aligned}\right\}$$

SISTEM

rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

npr. $M(p_1, p_2)$ je jedna stacionarna tačka.

$$A = \frac{\partial^2 z(p_1,p_2)}{\partial x^2}$$

$$B = \frac{\partial^2 z(p_1,p_2)}{\partial x \partial y}$$

$$C = \frac{\partial^2 z(p_1,p_2)}{\partial y^2}$$

$D = AC - B^2$

$D > 0$ f-ja ima ekstrem u tački $M(p_1, p_2)$

a) $A > 0$ imamo z_{min}

b) $A < 0$ imamo z_{max}

$D < 0$ f-ja nema ekstrem

$D = 0$ potrebno ispitati ponašanje f-je u okolini stacionarne tačke:

$\Delta z(M) = z(p_1 + \varepsilon, p_2 + \omega) - z(p_1, p_2)$ – priraštaj f-je

$\Delta z \geq 0 \quad \forall \varepsilon ; \forall \omega \Rightarrow$ u tački M f-ja ima minimum

$\Delta z \leq 0 \quad \forall \varepsilon ; \forall \omega \Rightarrow$ u tački M f-ja ima maksimum

(#) Naći ekstreme f-je $z = x^2 - 2x - y - \ln(2-y) + 4$.

Rj.

$D: 2-y>0$

$$\frac{\partial z}{\partial x} = 2x - 2$$

$$\frac{\partial z}{\partial y} = -1 - \frac{1}{2-y}\cdot(-1) = \frac{1}{2-y} - 1$$

$$2x-2=0$$
$$\frac{1}{2-y} - 1 = 0$$
$$x=1, \quad y=1$$

Tačka $M(1,1)$ je stacionarna tačka (kandidat za ekstrem)

$(2-y)^{-1}$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)(2-y)^{-2}\cdot(-1) = \frac{1}{(2-y)^2}$$

$M(1,1)$

$A=2,\ B=0,\ C=1$

$D = AC - B^2 = 2 > 0$

F-ja ima ekstrem.

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(1,1) = 1-2-1-\ln 1 + 4 = -2+4 = 2$$

(#) Naći ekstreme f-je $z = x^3 - 5xy + 5y^2 + 7x - 15y$.

R:

Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 3x^2 - 5y + 7$$

$$\frac{\partial z}{\partial y} = 10y - 5x - 15$$

$$3x^2 - 5y + 7 = 0 \quad | \cdot 2$$
$$-5x + 10y - 15 = 0$$

$$6x^2 - 10y + 14 = 0$$
$$-5x + 10y - 15 = 0 \quad +$$

$$6x^2 - 5x - 1 = 0$$

$$D = 25 + 24 = 49$$

$$x_{1,2} = \frac{5 \pm 7}{2 \cdot 6}$$

$$x_1 = \frac{-2}{2 \cdot 6} = \frac{-1}{6}, \quad x_2 = \frac{12}{12} = 1$$

$$6\left(x + \frac{1}{6}\right)(x - 1) = 0$$

Za $x_1 = -\frac{1}{6} \Rightarrow -5 \cdot \left(-\frac{1}{6}\right) + 10y - 15 = 0$

$$10y = 15 - \frac{5}{6}$$

$$10y = \frac{90 - 5}{6} = \frac{85}{6}$$

$$y = \frac{\cancel{85}^{17}}{\cancel{10}_{2} 6} = \frac{17}{12}$$

$x_2 = 1 \Rightarrow -5 + 10y - 15 = 0$

$$10y = 20$$

$$y = 2$$

Stacionarne tačke su $(1, 2)$ i $\left(-\frac{1}{6}, \frac{17}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = -5$$

$$\frac{\partial^2 z}{\partial y^2} = 10$$

Za $M_1(1,2)$

$A = 6, \ B = -5, \ C = 10, \ D = AC - B^2 = 60 - 25 > 0$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$z_{min}(1,2) = 1 - 10 + 20 + 7 - 30 = 8 + 10 - 30 = 8 - 20 = -12$$

Za $M_2\left(-\frac{1}{6}, \frac{17}{12}\right)$

$A = -1, \ B = -5, \ C = 10, \ D = AC - B^2 = -10 - 25 = -35$

F-ja u ovoj tački nema ekstrem

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj. Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}x = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}y = 1 - \frac{3y}{x^2+y^2+1}$$

Nađimo stacionarne tačke

$$1 - \frac{3x}{x^2+y^2+1} = 0$$

$$1 - \frac{3y}{x^2+y^2+1} = 0$$

$$3x = 3y \Rightarrow x = y$$

$$1 - \frac{3x}{2x^2+1} = 0 \quad / \cdot 2x^2+1$$

$$2x^2 + 1 - 3x = 0$$

$$2x^2 - 3x + 1 = 0$$

$$D = 9 - 8 = 1$$

$$x_{1,2} = \frac{3 \pm 1}{4}$$

$$x_1 = \frac{2}{4} = \frac{1}{2}$$

$$x_2 = 1$$

Stacionarne tačke su $M_1(\frac{1}{2}, \frac{1}{2})$ i $M_2(1,1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 0 - 3\cdot\frac{1\cdot(x^2+y^2+1) - x\cdot 2x}{(x^2+y^2+1)^2} = -3\cdot\frac{-x^2+y^2+1}{(x^2+y^2+1)^2} = 3\,\frac{x^2-y^2-1}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = 0 - 3x\cdot(-1)(x^2+y^2+1)^{-2}\cdot 2y = 6\,\frac{xy}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 - 3\,\frac{1\cdot(x^2+y^2+1) - y\cdot 2y}{(x^2+y^2+1)^2} = -3\,\frac{x^2-y^2+1}{(x^2+y^2+1)^2}$$

Za $M_1(\frac{1}{2}, \frac{1}{2})$, $A = 3\cdot\frac{-1}{(\frac{1}{2}+1)^2} = \frac{-3}{\frac{9}{4}} = \frac{-12}{9} = -\frac{4}{3}$, $B = \frac{2}{3}$, $C = -\frac{4}{3}$

$D = AC - B^2 = \frac{16}{9} - \frac{4}{9} > 0$ f-ja ima ekstrem u tački M_1

$A < 0$ f-ja ima minimum $z_{min}(\frac{1}{2}, \frac{1}{2}) = 1 - \frac{3}{2}\ln\frac{3}{2}$

Za $M_2(1,1)$, $A = -\frac{1}{3}$, $B = \frac{2}{3}$, $C = -\frac{1}{3}$

$D = AC - B^2 = \frac{1}{9} - \frac{4}{9} < 0$ f-ja u tački M_2 nema ekstrem

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj: Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{2x}{x^2+y^2+1} = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{2y}{x^2+y^2+1} = 1 - \frac{3y}{x^2+y^2+1}$$

Pronađimo stacionarne tačke

$$\left.\begin{array}{l} \frac{\partial z}{\partial x} = 0 \Rightarrow 1 = \frac{3x}{x^2+y^2+1} \\ \frac{\partial z}{\partial y} = 0 \Rightarrow 1 = \frac{3y}{x^2+y^2+1} \end{array}\right\} \Rightarrow x = y \text{ (dijeljenjem jednačina)}$$

Sad imamo $x=y$ i $1 = \frac{3x}{x^2+y^2+1} \Rightarrow 1 = \frac{3x}{2x^2+1} \Rightarrow 2x^2-3x+1=0$

$D = 9-8 = 1$

$x_1 = 1,\ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(1,1)$ i $M_2(\frac{1}{2},\frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2+y^2+1}\right)'_x = \frac{-3(x^2+y^2+1)+3x\cdot 2x}{(x^2+y^2+1)^2} = \frac{3x^2-3y^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2+y^2+1}\right)'_y = \left|\begin{array}{l}\text{zbog}\\ \text{simetričnosti}\end{array}\right| = \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x\cdot 2y}{(x^2+y^2+1)^2} = \frac{6xy}{(x^2+y^2+1)^2}$$

Za tačku $M_1(1,1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$

$D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstrema

Za tačku $M_2(\frac{1}{2},\frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$

$B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A < 0 \Rightarrow$ u M_2 f-ja ima maximum. $z_{max}(\frac{1}{2},\frac{1}{2}) = \frac{1}{2}+\frac{1}{2} - \frac{3}{2}\ln(\frac{1}{4}+\frac{1}{4}+1) = 1 - \ln\frac{27}{8}$

(#) Naći ekstreme f-je $z = \frac{8}{x} + \frac{x^2}{y} + y + 1.$

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 8\cdot(-1)\,x^{-2} + 2\,\frac{x}{y} = \frac{-8}{x^2} + 2\,\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x^2\cdot(-1)\,y^{-2} + 1 = \frac{-x^2}{y^2} + 1$$

$$-\frac{8}{x^2} + \frac{2x}{y} = 0$$

$$-\frac{x^2}{y^2} + 1 = 0$$

$$\frac{8}{x^2} - 2\,\frac{x}{y} = 0$$

$$\frac{x^2}{y^2} = 1 \Rightarrow \left(\frac{x}{y}\right)^2 = 1$$

Prema tome $\frac{x}{y} = 1$ i $\frac{x}{y} = -1$

Za $\frac{x}{y} = 1 \Rightarrow \frac{8}{x^2} - 2\cdot 1 = 0$

$$\frac{8}{x^2} = 2 \quad /\cdot x^2 \ (x \neq 0)$$

$2x^2 = 8$

$x^2 = 4$

$x_1 = -2,\ x_2 = 2$

$x_1 = -2 \Rightarrow \frac{x}{y} = 1$

$y = -2$

$(-2, -2)$

za $x_2 = 2 \Rightarrow$

$\Rightarrow \frac{x}{y} = 1$

$y_2 = 2$

$(2, 2)$

Za $\frac{x}{y} = -1$ imamo

$$\frac{8}{x^2} + 2 = 0$$

$$\frac{8}{x^2} = -2 \quad /\cdot x^2 \ (x \neq 0)$$

$-2x^2 = 8$

ova jednačina nema rješenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(2,2)$.

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = (-8)(-2)\,x^{-3} + \frac{2}{y} = \frac{16}{x^3} + \frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\cdot(-1)\,y^{-2} = \frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\cdot(-2)\,y^{-3} = \frac{2x^2}{y^3}$$

Za $M_1(-2,-2)$

$A = \frac{16}{-8} + \frac{2}{-2} = -2 - 1 = -3$

$B = \frac{-2\cdot(-2)}{4} = 1,\quad C = \frac{2\cdot 4}{-8} = -1$

$D = AC - B^2 = 3 - 1 = 2 > 0$

f-ja u tački $M_1(-2,-2)$ ima ekstrem,

$A < 0$ f-ja ima maksimum

$z_{max}(-2,-2) = -4 - 2 - 2 + 1 = -7$

Za $M_2(2,2)$

$A = 2 + 1 = 3,\quad B = \frac{-4}{4} = -1,\quad C = \frac{8}{8} = 1$

$D = AC - B^2 = 3 - 1 = 2 > 0$ f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2) = 4 + 2 + 2 + 1 = 9$

(#) Naći ekstreme f-je $z=(x^2+y)\sqrt{e^y}$.

Rj. $\frac{\partial z}{\partial x} = 2x\sqrt{e^y}$

$$\frac{\partial z}{\partial y} = \sqrt{e^y} + (x^2+y)\frac{1}{2\sqrt{e^y}}\cdot e^y = \sqrt{e^y} + (x^2+y)\cdot\frac{1}{2}\sqrt{e^y}$$

$$= \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\sqrt{e^y} = \frac{1}{2}(x^2+y+2)\sqrt{e^y}$$

$$2x\sqrt{e^y} = 0$$

$$\frac{1}{2}(x^2+y^2+1)\sqrt{e^y} = 0$$

$e^y > 0 \ \forall y \in \mathbb{R}$

prema tome $x=0$

$\sqrt{e^y} > 0 \ \forall x \in \mathbb{R}$

$x^2+y+2=0$

$x=0 \Rightarrow y+2=0$

$y=-2$

$M(0,-2)$ je stacionarna tačka (kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\frac{1}{2\sqrt{e^y}}\cdot e^y = x\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\sqrt{e^y} + \left(\frac{1}{2}x^2+\frac{1}{2}y+1\right)\frac{e^y\cdot\sqrt{e^y}}{2\sqrt{e^y}\cdot\sqrt{e^y}} = \frac{1}{2}\sqrt{e^y}\left(\frac{1}{2}x^2+\frac{1}{2}y+2\right)$$

$M(0,-2)$

$A = 2\sqrt{e^{-2}} = 2\frac{1}{\sqrt{e^2}}$

$B=0$

$C = \frac{1}{2}\sqrt{e^{-2}}\left(\frac{1}{2}0 + \underbrace{\frac{1}{2}(-2)}_{-1} + 2\right) = \frac{1}{2}\sqrt{\frac{1}{e^2}}$

$D = AC - B^2 = \frac{2}{\sqrt{e^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{e^2}} = \frac{1}{e^2}$

$D>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(0,-2) = (0-2)\sqrt{e^{-2}} = (-2)\cdot\frac{1}{\sqrt{e^2}} \approx -0.7358$$

(#) Naći ekstreme f-je $z=e^{-2x^2}(x-y^2)$.

Rj. Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x}=e^{-2x^2}\cdot(-4)x(x-y^2)+e^{-2x^2}\cdot 1=e^{-2x^2}(-4x^2+4xy^2+1)$$

$$\frac{\partial z}{\partial y}=e^{-2x^2}\cdot(-2)y=-2ye^{-2x^2}$$

$$e^{-2x^2}(-4x^2+4xy^2+1)=0$$
$$-2y\,e^{-2x^2}=0$$

e^{-2x^2} je uvijek pozitivno

$$-4x^2+4xy^2+1=0$$
$$-2y=0 \Rightarrow y=0$$

$$-4x^2+1=0$$
$$x^2=\frac{1}{4} \Rightarrow x_1=-\frac{1}{2},\; x_2=\frac{1}{2}$$

Stacionarne tačke su $M_1(-\frac{1}{2},0)$ i $M_2(\frac{1}{2},0)$

$$\frac{\partial^2 z}{\partial x^2}=e^{-2x^2}\cdot(-4x)(-4x^2+4xy^2+1)+e^{-2x^2}(-8x+4y^2)=$$
$$=e^{-2x^2}(16x^3-16x^2y^2-4x-8x+4y^2)=e^{-2x^2}(16x^3-16x^2y^2-12x+4y^2)$$
$$=4e^{-2x^2}(4x^3-4x^2y^2-3x+y^2)$$

$$\frac{\partial^2 z}{\partial x\partial y}=e^{-2x^2}(8xy)=8xye^{-2x^2}$$

$$\frac{\partial^2 z}{\partial y^2}=-2e^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2},0)$

$$A=4e^{-2\cdot\frac{1}{4}}\left(4\cdot(-\tfrac{1}{8})-4\cdot\tfrac{1}{4}\cdot 0-3(-\tfrac{1}{2})+0\right)=4e^{-\frac{1}{2}}\left(-\tfrac{1}{2}+\tfrac{3}{2}\right)=\frac{4}{\sqrt{e}}$$

$B=0$, $C=-2e^{-2\cdot\frac{1}{4}}=-2e^{-\frac{1}{2}}=\frac{-2}{\sqrt{e}}$

$D=AC-B^2=\frac{-8}{e}<0$

F-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2},0)$

$$A=4e^{-2\cdot\frac{1}{4}}\left(4\cdot\tfrac{1}{8}-0-3\cdot\tfrac{1}{2}+0\right)=$$
$$=4e^{-\frac{1}{2}}\left(\tfrac{1}{2}-\tfrac{3}{2}\right)=\frac{-4}{\sqrt{e}}$$

$B=0$, $C=-2e^{-2\cdot\frac{1}{4}}=\frac{-2}{\sqrt{e}}$

$D=AC-B^2=\frac{8}{e}>0 \Rightarrow$ f-ja z u tački M_2 ima ekstrem

$A<0 \Rightarrow z_{max}(\frac{1}{2},0)=e^{-2\cdot\frac{1}{4}}(\frac{1}{2}-0)=\frac{1}{2}\cdot e^{-\frac{1}{2}}=\frac{1}{2\sqrt{e}}$

#) Odrediti ekstremne vrijednosti f-je

$$z = \frac{xy}{2} + (47 - x - y)\left(\frac{x}{3} + \frac{y}{4}\right).$$

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{2}y + (-1)\left(\frac{x}{3} + \frac{y}{4}\right) + (47 - x - y)\cdot\frac{1}{3} = \frac{1}{2}y - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{3} - \frac{1}{3}x - \frac{1}{3}y$$

$$= -\frac{2}{3}x + \frac{6-3-4}{12}y + \frac{47}{3} = -\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}x + (-1)\left(\frac{x}{3} + \frac{y}{4}\right) + (47 - x - y)\cdot\frac{1}{4} = \frac{1}{2}x - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{4} - \frac{1}{4}x - \frac{1}{4}y$$

$$= -\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4}$$

$$-\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3} = 0 \quad /\cdot 12$$

$$-\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4} = 0 \quad /\cdot 12$$

$$-8x - y + 188 = 0$$

$$-x - 6y + 141 = 0$$

$$-8x - y + 188 = 0$$

$$x = -6y + 141$$

$$-8(-6y + 141) - y + 188 = 0$$

$$48y - 1128 - y + 188 = 0$$

$$47y = 940$$

$$y = 20$$

$$x = -6y + 141 = -120 + 141 = 21$$

Stacionarna tačka je $M(21,20)$.

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{12}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{2}$$

$D = AC - B^2$

$M(21,20)$

$A = -\frac{2}{3}, \quad B = -\frac{1}{12}, \quad C = -\frac{1}{2}$

$$D = \frac{2}{6} - \frac{1}{144} = \frac{1}{3} - \frac{1}{144} > 0$$

f-ja z ima ekstrem

$A < 0$ f-ja ima maksimum

$$z_{max}(21,20) = 21\cdot 10 + (47 - 41)(7 + 5) = 210 + 6\cdot 12 = 210 + 72 = 282$$

$z_{max}(21,20) = 282$ traženi ekstrem f-je

Nađi ekstreme f-je $z=x^4+y^4-2x^2$.

Rj: $\frac{\partial z}{\partial x}=4x^3-4x$ $\qquad$ $\begin{array}{l} 4x^3-4x=0 \quad /:4 \\ 4y^3=0 \quad /:4 \\ \hline x^3-x=0 \\ y^3=0 \\ \hline \end{array}$ $\qquad$ $\begin{array}{l} x(x^2-1)=0 \\ y^3=0 \\ \hline x(x-1)(x+1)=0 \\ y^3=0 \\ \hline \end{array}$

$\frac{\partial z}{\partial y}=4y^3$

$y=0 \ \wedge \ (x_1=0,\ x_2=1,\ x_3=-1)$

Stacionarne tačke f-je su $M_1(-1,0)$, $M_2(0,0)$ i $M_3(1,0)$.

$\frac{\partial^2 z}{\partial x^2}=12x^2-4$ $\qquad$ za $M_1(-1,0)$, $A=8$, $B=0$, $C=0$

$\frac{\partial^2 z}{\partial x \partial y}=0$ $\qquad$ $D=0$ ispitujemo ponašanje f-je u okolini tačke $M_1(-1,0)$

$\frac{\partial^2 z}{\partial y^2}=12y^2$ $\qquad$ $\Delta z = z(-1+\varepsilon, 0+\omega)-z(-1,0)=$

$=(-1+\varepsilon)^4+\omega^4-2(-1+\varepsilon)^2-\left[(-1)^4+0^4-2(-1)^2\right]$

paskalov trougao

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$=1-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4+\omega^4-2(1-2\varepsilon+\varepsilon^2)-(1-2)$

$=1-4\varepsilon+6\varepsilon^2-4\varepsilon^3+\varepsilon^4+\omega^4-2+4\varepsilon-2\varepsilon^2+1$

$=\varepsilon^4-4\varepsilon^3+4\varepsilon^2+\omega^4=\varepsilon^2(\varepsilon^2-4\varepsilon+4)+\omega^4$

$=\varepsilon^2(\varepsilon-2)^2+\omega^4 \quad \geq 0$ za $\forall\varepsilon$ i $\forall\omega$

f-ja ima minimum u tački $M_1(-1,0)$, $z_{min}=-1$

za $M_2(0,0)$, $A=-4$, $B=0$, $C=0$, $D=AC-B^2=0$

ispitujemo ponašanje f-je u okolini tačke

$\Delta z = z(0+\varepsilon, 0+\omega)-z(0,0)=\varepsilon^4+\omega^4-2\varepsilon^2=\varepsilon^2(\varepsilon^2-2)+\omega^4$

$\varepsilon=0$: $\quad \Delta z=\omega^4$

$\omega=0$: $\quad \Delta z=\varepsilon^2(\varepsilon^2-2) \Rightarrow \Delta z<0$ za $\varepsilon^2<2$

$\Delta z>0$ za $\varepsilon^2>2$

Priraštaj f-je je promjenjivog znaka pa f-ja nema ekstrem u tački M_2

za $M_3(1,0)$, $A=8$, $B=0$, $C=0$, $D=AC-B^2=0$ ispitujemo ponašanje f-je u okolini tačke

$\Delta z = z(1+\varepsilon, 0+\omega)-z(1,0)=(1+\varepsilon)^4+\omega^4-2(1+\varepsilon)^2-(1-2)$

$=1+4\varepsilon+6\varepsilon^2+4\varepsilon^3+\varepsilon^4+\omega^4-2-4\varepsilon-2\varepsilon^2+1=\varepsilon^4+4\varepsilon^3+4\varepsilon^2+\omega^4$

$\Delta z=\varepsilon^2(\varepsilon+2)^2+\omega^4 \geq 0 \quad \forall\varepsilon$ i $\forall\omega$ $\qquad$ f-ja z u tački M_3 ima min

$z_{min}=-1$

(#) Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x}=4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x)=(4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y}=6y e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y)=(6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \quad \forall(x,y\in\mathbb{R})$

$x=0$; $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$

$y^2=1$

$y_{1,2}=\pm 1$

$M_2(0,-1)$

$M_3(0,1)$

ili

$2-2x^2-3y^2=0$; $y=0$

$2x^2=2$

$x^2=1$

$x_{1,2}=\pm 1$

$M_4(-1,0)$

$M_5(1,0)$

ili

$$\begin{array}{r} 2-2x^2-3y^2=0\\ -\ \ 3-2x^2-3y^2=0\\ \hline -1=0 \end{array}$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2}=(4-12x^2-6y^2)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x)=(8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=(-12xy)e^{-x^2-y^2}+(4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y)=(-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(6-4x^2-18y^2)e^{-x^2-y^2}+(6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y)=(-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

(#) Naći stacionarne tačke f-je $z = xy\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = y\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2x = y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = x\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2y = x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2}$$

$$y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2} = 0$$

$$x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2} = 0$$

$$y\left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right) = 0$$

$$x\left(\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right) = 0$$

$y=0$ ili $\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$

$x=0$ ili $\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$

$y=0$ i $x=0$ $\quad M_1(0,0)$

za M_1 f-ja nije definisana

ili

$y=0$ i $\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$

$$\ln x^2 = 0$$

$$x^2 = 1$$

$$x_1 = -1, \quad x_2 = 1$$

$M_2(-1,0)$, $M_3(1,0)$

ili

$\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$ i $x=0$

$$\ln y^2 = 0$$

$$y_1 = -1 \quad y_2 = 1$$

$M_4(0,-1)$

$M_5(0,1)$

ili

$$\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0 \quad (1)$$

$$\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0 \quad (2)$$

$(1)-(2)$: $\quad \frac{2x^2}{x^2+y^2} - \frac{2y^2}{x^2+y^2} = 0$

$$2x^2 - 2y^2 = 0$$

$$2(x^2-y^2) = 0$$

$$2(x-y)(x+y) = 0$$

$x = y$ ili $x = -y$.

za $x=y$: $\ln(2x^2) + 1 = 0$

$$\ln(2x^2) = -1$$

$$e^{-1} = 2x^2$$

$$x^2 = \frac{1}{2e}$$

$$x_{1,2} = \pm\frac{1}{\sqrt{2e}}$$

$M_6\left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)$, $M_7\left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

za $y=-x$: $\ln(2x^2) + 1 = 0$

$$x_{1,2} = \pm\frac{1}{\sqrt{2e}}$$

$M_8\left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}}\right)$

$M_9\left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right)$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8$ i M_9.

Ekstremi f-ja tri promjenjive

Neka je data f-ja $u = f(x, y, z)$

$$\left.\begin{aligned} u'_x &= 0 \\ u'_y &= 0 \\ u'_z &= 0 \end{aligned}\right\}$$ rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

SISTEM

npr. $M(p_1, p_2, p_3)$ je jedna stacionarna tačka

I način: Silvesterov kriterij

$a_{11} = u''_{xx}(M)$ $\quad a_{21} = u''_{yx}(M)$ $\quad a_{31} = u''_{zx}(M)$

$a_{12} = u''_{xy}(M)$ $\quad a_{22} = u''_{yy}(M)$ $\quad a_{32} = u''_{zy}(M)$

$a_{13} = u''_{xz}(M)$ $\quad a_{23} = u''_{yz}(M)$ $\quad a_{33} = u''_{zz}(M)$

$$T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a) $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, $\det(T) > 0 \Rightarrow M$ je tačka minimuma

b) $a_{11} < 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, $\det(T) < 0 \Rightarrow M$ je tačka maksimuma

c) $a_{11} = 0$ ili $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$ ili $\det T = 0 \Rightarrow$ potrebno je ispitati f-ju u okolini tačke M (formira se priraštaj Δu, $\Delta u = u(p_1+\varepsilon, p_2+\omega, p_3+\delta) - u(p_1, p_2, p_3)$)

d) za sve ostale kombinacije znakova u tački M nema ekstrema.

II način: Pomoću diferencijala d^2u

a) Ako je $d^2u(M) > 0$ tada u tački M f-ja ima min

b) Ako je $d^2u(M) < 0$ tada u tački M f-ja ima max

c) $d^2u(M)$ promjenjivog znaka $\Rightarrow$ u tački M f-ja nema ekstrema

(#) Nađi ekstreme f-je $u = \sin x + \sin y + \sin z - \sin(x+y+z)$
gdje su $0 \le x \le \pi$, $0 \le y \le \pi$ i $0 \le z \le \pi$.

Rj.
$$u'_x = \cos x - \cos(x+y+z)$$
$$u'_y = \cos y - \cos(x+y+z)$$
$$u'_z = \cos z - \cos(x+y+z)$$

$$\cos x - \cos(x+y+z) = 0$$
$$\cos y - \cos(x+y+z) = 0$$
$$\underline{\cos z - \cos(x+y+z) = 0}$$
$$\cos x = \cos y = \cos z = \cos(x+y+z)$$

Možemo li imati dva različita ugla koja nisu u istom kvadrantu (prvom ili drugom) a da su im kosinusi jednaki.

$x = y = z$

$\cos x - \cos 3x = 0 \quad \Rightarrow \quad \cos x - \cos 3x = (-2)\sin(-x)\sin(2x) = 2\sin(x)\sin(2x) = 0$

$$\left.\begin{aligned} \cos x &= \cos\frac{x-3x+x+3x}{2} = \cos\left(\frac{x-3x}{2} + \frac{x+3x}{2}\right) = \cos(-x)\cos(2x) - \sin(-x)\sin(2x) \\ \cos 3x &= \cos\frac{x+3x-x+3x}{2} = \cos\left(\frac{x+3x}{2} - \frac{x-3x}{2}\right) = \cos(2x)\cos(-x) + \sin(2x)\sin(-x) \end{aligned}\right\} -$$

$\sin 2x = 0$ ili $\sin x = 0 \quad \Rightarrow \quad 2x = 0$ ili $2x = \pi$ ili $x = 0$ ili $x = \pi$

$0 \le x \le \pi \qquad x = 0$ ili $x = \frac{\pi}{2}$ ili $x = \pi$

Stacionarne tačke su $M_1(0,0,0)$, $M_2(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ i $M_3(\pi, \pi, \pi)$

$$u''_{xx} = -\sin x + \sin(x+y+z) \qquad u''_{yx} = \sin(x+y+z) \qquad u''_{zx} = \sin(x+y+z)$$
$$u''_{xy} = \sin(x+y+z) \qquad u''_{yy} = -\sin y + \sin(x+y+z) \qquad u''_{zy} = \sin(x+y+z)$$
$$u''_{xz} = \sin(x+y+z) \qquad u''_{yz} = \sin(x+y+z) \qquad u''_{zz} = -\sin z + \sin(x+y+z)$$

za tačku $M_1(0,0,0)$

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ispitujemo f-ju u okolini tačke M_1

$$\Delta u = u(0+\varepsilon, 0+\omega, 0+\delta) - u(0,0,0) = \sin\varepsilon + \sin\omega + \sin\delta - \sin(\underbrace{\varepsilon+\omega}_{\alpha} + \underbrace{\delta}_{\beta})$$
$$= \sin\varepsilon + \sin\omega + \sin\delta - (\sin(\varepsilon+\omega)\cos\delta + \sin\delta\cos(\varepsilon+\omega)) =$$
$$= \underline{\sin\varepsilon} + \underline{\sin\omega} + \sin\delta - \underline{\cos\delta(\sin\varepsilon\cos\omega} - \underline{\sin\omega\cos\varepsilon)}$$
$$- \sin\delta(\cos\varepsilon\cos\omega - \sin\varepsilon\sin\omega) = \sin\varepsilon(1 - \cos\delta\cos\omega) + \sin\omega(1 - \cos\delta\cos\varepsilon)$$
$$+ \sin\delta(1 - \cos\varepsilon\cos\omega) + \sin\varepsilon\sin\omega\sin\delta \ge 0 \quad \forall\varepsilon\ \forall\omega\ \forall\delta$$

Ove iste vrijednosti dobijemo i za tačku $M_3(\pi, \pi, \pi)$

$\Rightarrow$ f-ja u tački M_1 i M_3 ima minimum, $u_{min} = 0$

za tačku $M_2(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$

$$T = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \qquad a_{11} < 0, \quad \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0, \quad \det T = \begin{vmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{vmatrix} \begin{array}{c} \underline{\underline{I_v - III_v \cdot 2}} \\ II_v - III_v \end{array} \begin{vmatrix} 0 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & -1 & -2 \end{vmatrix}$$

$\det T = -4 \Rightarrow$ U tački f-ja ima maksimum, $u_{max} = 4$

Naći ekstreme f-je $u = x^2 + y^2 + z^2 + 2x + 4y - 6z$.

R: $u'_x = 2x + 2$ $2x + 2 = 0$

$u'_y = 2y + 4$ $2y + 4 = 0$

$u'_z = 2z - 6$ $2z - 6 = 0$

$x = -1, \; y = -2, \; z = 3$

Tačka $M(-1, -2, 3)$ je stacionarna tačka.

$u''_{xx} = 2$ $u''_{yx} = 0$ $u''_{zx} = 0$

$u''_{xy} = 0$ $u''_{yy} = 2$ $u''_{zy} = 0$

$u''_{xz} = 0$ $u''_{yz} = 0$ $u''_{zz} = 2$

$$T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$

i $\det T = 8 > 0 \Rightarrow$ u tački M f-ja ima minimum

$z_{min} = 1 + 4 + 9 - 2 - 8 - 18 = -14$

Uslovni ekstremi f-ja dviju promjenjivih

Ako trebamo naći ekstrem f-je $z=f(x,y)$ tako da x i y zadovoljavaju neki uslov $g(x,y)=0$ tada tražimo ekstrem Lagranžove f-je $F(x,y;\lambda)=f(x,y)+\lambda g(x,y)$.

$$\frac{\partial F}{\partial x}=0$$
$$\frac{\partial F}{\partial y}=0$$
$$\frac{\partial F}{\partial z}=0$$

SISTEM

rješavanjem sistema dobićemo neke stacionarne tačke i dalji proces se nastavlja kao kod traženja ekstrema f-je dvije promjenjive

II način: neka je $M(p_1,p_2)$ neka stacionarna tačka

$$d^2F(p_1,p_2)=F''_{xx}(p_1,p_2)\,dx^2+2F''_{xy}(p_1,p_2)\,dx\,dy+F''_{yy}(p_1,p_2)\,dy^2$$

$$d^2F(p_1,p_2)>0 \Rightarrow z_{min}(p_1,p_2)$$
$$d^2F(p_1,p_2)<0 \Rightarrow z_{max}(p_1,p_2)$$

Ako se desi slučaj da imamo više uslova, onda uvodimo više parametara $(\lambda,\mu,\dots)$.

1. Naći ekstreme f-je $z=6-4x-3y$ uz uslov $x^2+y^2=1$.

Rj. $F(x,y)=6-4x-3y+\lambda(x^2+y^2-1)\neq 0$

$$\frac{\partial F}{\partial x}=-4+2\lambda x$$
$$\frac{\partial F}{\partial y}=-3+2\lambda y$$
$$\frac{\partial F}{\partial \lambda}=x^2+y^2-1$$

$$\begin{aligned}2\lambda x-4&=0\\2\lambda y-3&=0\\x^2+y^2-1&=0\end{aligned}$$

$$\begin{aligned}2\lambda x&=4\\2\lambda y&=3\\x^2+y^2&=1\end{aligned}$$

$$x=\frac{2}{\lambda}$$
$$y=\frac{3}{2\lambda}$$
$$x^2+y^2=1$$
$$\frac{4}{\lambda^2}+\frac{9}{4\lambda^2}=1$$
$$\frac{25}{4\lambda^2}=1$$

$$4\lambda^2=25$$
$$\lambda_{1,2}=\pm\frac{5}{2}$$

$$\lambda_1=-\frac{5}{2} \Rightarrow x_1=-\frac{4}{5}\ ;\ y_1=\frac{3}{2\cdot(-\frac{5}{2})}=-\frac{3}{5}$$

$$\lambda_2=\frac{5}{2} \Rightarrow x_2=\frac{2}{\frac{5}{2}}=\frac{4}{5}\ ;\ y_2=\frac{3}{2\cdot\frac{5}{2}}=\frac{3}{5}$$

Stacionarne tačke su $M(-\frac{4}{5},-\frac{3}{5})$ za $\lambda=-\frac{5}{2}$ i $N(\frac{4}{5},\frac{3}{5})$ za $\lambda=\frac{5}{2}$.

$\frac{\partial^2 F}{\partial x^2} = 2\lambda$ za $M(-\frac{4}{5}, -\frac{3}{5}), \lambda = -\frac{5}{2}$

$A = -5, \; B = 0, \; C = -5, \; D = AC - B^2 = 25 > 0$

$\frac{\partial^2 F}{\partial x \partial y} = 0$ f-ja ima ekstrem, $A < 0$ f-ja ima maksimum

$\frac{\partial^2 F}{\partial y^2} = 2\lambda$ $z_{max}(-\frac{4}{5}, -\frac{3}{5}) = 6 - 4(-\frac{4}{5}) - 3(-\frac{3}{5}) = \frac{30+16+9}{5} = \frac{55}{5} = 11$

za $N(\frac{4}{5}, \frac{3}{5}), \lambda = \frac{5}{2}$, $A = 5, B = 0, C = 5, D = AC - B^2 = 25 > 0$

f-ja ima ekstrem u tački N, $A > 0$ f-ja ima minimum

$z_{min}(\frac{4}{5}, \frac{3}{5}) = 6 - 4 \cdot \frac{4}{5} - 3 \cdot \frac{3}{5} = \frac{30-16-9}{5} = \frac{5}{5} = 1$

2. Nađi uslovne ekstreme f-je $z = y + 2x + 3$ uz uslov $x^2 - 6x + y + 5 = 0$.

Rj. $F(x,y) = 2x + y + 3 + \lambda(x^2 - 6x + y + 5)$

$\frac{\partial F}{\partial x} = 2 + 2\lambda x - 6\lambda$

$\frac{\partial F}{\partial y} = 1 + \lambda$

$\frac{\partial F}{\partial \lambda} = x^2 - 6x + y + 5$

$$\begin{aligned} 2\lambda x - 6\lambda + 2 &= 0 \quad /:2 \\ \lambda + 1 &= 0 \\ x^2 - 6x + y + 5 &= 0 \\ \hline \lambda x &= 3\lambda - 1 \\ \lambda &= -1 \\ x^2 - 6x + y + 5 &= 0 \\ \hline \end{aligned}$$

$$\begin{aligned} -x &= -3 - 1 \\ x &= 4 \\ x^2 - 6x + y + 5 &= 0 \\ 16 - 24 + y + 5 &= 0 \\ y &= 3 \end{aligned}$$

Tačka $M(4,3)$ je stacionarna tačka, za $\lambda = -1$

$\frac{\partial^2 F}{\partial x^2} = 2\lambda$ $M(4,3), \lambda = -1$

$A = -2, \; B = 0, \; C = 0 \Rightarrow D = AC - B^2 = 0$

$\frac{\partial^2 F}{\partial x \partial y} = 0$ $d^2F = \frac{\partial^2 F}{\partial x^2} dx^2 + \frac{\partial^2 F}{\partial x \partial y} \cancel{dxdy} + \frac{\partial^2 F}{\partial y^2} dy^2$

$\frac{\partial^2 F}{\partial y^2} = 0$ $d^2F = 2\lambda \, dx^2 \Rightarrow d^2F = -2\,dx^2 < 0$

U tački $M(4,3)$ f-ja ima maksimum, $z_{max}(4,3) = 3 + 8 + 3 = 14$

3. Odrediti ekstreme f-je $z = x^2 + y^2$ uz uslov $\frac{x}{2} + \frac{y}{3} = 1$.

Rj. $z_{min}(\frac{18}{13}, \frac{12}{13}) = \frac{36}{13}$, $\lambda = -\frac{72}{13}$

4. Nađi uslovne ekstreme f-je $z = \ln(x+y)$, ako je $x^2 + 2y^2 = 4$.

Rj. $z_{max}(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}) = \ln(3\sqrt{\frac{2}{3}})$, $\lambda = -\frac{1}{8}$

(#) Naći uslovne ekstreme f-je $z=2x^4+8y^4+24$ ako je $8x+4y=1$.

Rj. $F(x,y,\lambda)=2x^4+8y^4+24+\lambda(8x+4y-1)$

$$\frac{\partial F}{\partial x}=8x^3+8\lambda$$

$$\frac{\partial F}{\partial y}=32y^3+4\lambda$$

$$\frac{\partial F}{\partial \lambda}=8x+4y-1$$

$8x^3+8\lambda=0 \quad /:8$

$32y^3+4\lambda=0 \quad /:4$

$x^3+\lambda=0$

$-\ 8y^3+\lambda=0$

$x^3-8y^3=0$

$x^3=8y^3$

$x=2y$

$8x+4y-1=0$

$8\cdot 2y+4y-1=0$

$20y=1$

$y=\frac{1}{20}$

$x=2\cdot\frac{1}{20}=\frac{1}{10}$.

$M_1(\frac{1}{10},\frac{1}{20})$ je stacionarna tačka

$$\frac{\partial^2 F}{\partial x^2}=24x^2$$

$$\frac{\partial^2 F}{\partial x\partial y}=0$$

$$\frac{\partial^2 F}{\partial y^2}=96y^2$$

$D=AC-B^2$

$M_1(\frac{1}{10},\frac{1}{20})$

$A=24\cdot\frac{1}{100}=\frac{24}{100}=\frac{12}{50}=\frac{6}{25}$

$B=0$

$C=\cancel{96}^{\cancel{48}\ 24}\cdot\frac{1}{\cancel{20}_{10}\cdot\cancel{20}_{10}}=\frac{24}{100}=\frac{12}{50}=\frac{6}{25}$

$D=\left(\frac{6}{25}\right)^2>0$ f-ja ima ekstrem

$A>0$ f-ja ima minimum

$Z_{min}(\frac{1}{10},\frac{1}{20})=2\cdot\frac{1}{10^4}+8\cdot\frac{1}{20^4}+24=\frac{2}{10\,000}+\frac{\cancel{8}^{\cancel{4}\ \cancel{2}\ 1}}{\cancel{20}_{10}\cdot\cancel{20}_{10}\cdot\cancel{20}_{10}\cdot 20}+24$

$=\frac{2}{10\,000}+\frac{1}{20\,000}+\frac{24\,000}{1\,000}=\frac{4+1+480\,000}{20\,000}=$

$=\frac{480\,005}{20\,000}\,\frac{:5}{:5}=\frac{96\,001}{4\,000}$

$Z_{min}=\frac{96\,001}{4000}$ je minimum f-je u tački $M(\frac{1}{10},\frac{1}{20})$

\# Naći uslovne ekstreme f-je $z=(x-y)^4+1$ ako je $x^2+y^2=18$.

Rj. $F(x,y,\lambda)=(x-y)^4+1+\lambda(x^2+y^2-18)$

$$\frac{\partial F}{\partial x}=4(x-y)^3+2\lambda x$$

$$\frac{\partial F}{\partial y}=4(x-y)^3\cdot(-1)+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-18$$

$$4(x-y)^3+2\lambda x=0 \quad \dots(1)$$
$$-4(x-y)^3+2\lambda y=0 \quad \dots(2)$$
$$x^2+y^2-18=0 \quad \dots(3)$$

(1) (2): $2\lambda x+2\lambda y=0 \quad /:2$

$\lambda(x+y)=0$

$\lambda=0$ ili $x+y=0$

a) $x+y=0$

$x=-y$

$(-y)^2+y^2=18$

$2y^2=18$

$y_{1,2}=\pm 3$

$y_1=-3 \Rightarrow x_1=3$

$y_2=3 \Rightarrow x_2=-3$

$M_1(3,-3)$, $M_2(-3,3)$

za M_1 (1) $\Rightarrow 6\lambda=-4\cdot 6^3$

za M_2 (1) $\Rightarrow -6\lambda=-4\cdot(-6)^3$

$\Rightarrow \lambda=-144$

b) $\lambda=0$

(1) $\Rightarrow 4(x-y)^3=0$

$x=y$

$2y^2=18$

$y_{3,4}=\pm 3$

$M_3(-3,-3)$

$M_4(3,3)$

$\lambda=0$

Stacionarne tačke su $M_1(3,-3)$ $M_2(-3,3)$ za $\lambda=-144$ i $M_3(-3,-3)$ i $M_4(3,3)$ za $\lambda=0$.

$$\frac{\partial^2 F}{\partial x^2}=12(x-y)^2+2\lambda$$

$$\frac{\partial F}{\partial x\partial y}=-12(x-y)^2$$

$$\frac{\partial F}{\partial y^2}=12(x-y)^2+2\lambda$$

$D=AC-B^2$

$M_1(3,-3)$, $\lambda=-144$

$A=12\cdot 36-2\cdot 144=144$

$B=-12\cdot 36=-432$

$C=144$

$D=20\,736-186\,624$

f-ja u tački M_1 nema ekstrema

$M_2(-3,3)$, $\lambda=-144$

$\left.\begin{array}{l} A=144 \\ B=432 \\ C=144 \end{array}\right\} D<0$

f-ja u tački M_2 nema ekstrem

$M_3(-3,-3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_3(-3,-3)$

$$\Delta z(M_3)=z(-3+\varepsilon,-3+\omega)-z(-3,-3)=(-3+\varepsilon+3-\omega)^4+1-1=(\varepsilon-\omega)^4>0 \quad \forall\varepsilon \text{ i } \forall\omega$$

Priraštaj f-je u okolini tačke M_3 je pozitivan pa f-ja u M_3 ima minimum, $z_{min}(-3,-3)=1$

$M_4(3,3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_4(3,3)$

$$\Delta z(M_4)=z(3+\varepsilon,3+\omega)-z(3,3)=(3+\varepsilon-3-\omega)^4+1-1=(\varepsilon-\omega)^4>0 \quad \forall\varepsilon \text{ i } \forall\omega$$

Priraštaj f-je u okolini tačke M_4 je pozitivan pa f-ja u M_4 ima minimum, $z_{min}(3,3)=1$.

(#) Naći uslovne ekstreme f-je $z = 2x + 4y$ ako je
$\frac{2}{x} + \frac{4}{y} = 3.$

Rj. Formirajmo Lagranžovu f-ju $F(x,y,\lambda) = 2x + 4y + \lambda\left(\frac{2}{x} + \frac{4}{y} - 3\right).$

$\frac{\partial F}{\partial x} = 2 + 2\lambda \cdot \frac{(-1)}{x^2}$ $\quad \left(\frac{1}{x}\right)' = (x^{-1})' = (-1)(x^{-2})$ $\quad (x^{-2})' = (-2)x^{-3} = \frac{-2}{x^3}$

$\frac{\partial F}{\partial y} = 4 + 4\lambda \cdot \frac{(-1)}{y^2}$

Formirajmo sistem
$$4 - \frac{4\lambda}{y^2} = 0 \quad /:4$$
$$2 - \frac{2\lambda}{x^2} = 0 \quad /:2$$
$$\frac{2}{x} + \frac{4}{y} = 3$$

$\frac{\partial F}{\partial \lambda} = \frac{2}{x} + \frac{4}{y} - 3$

$1 - \frac{\lambda}{x^2} = 0 \qquad 1 = \frac{\lambda}{x^2} \quad (1)$

$1 - \frac{\lambda}{y^2} = 0 \qquad 1 = \frac{\lambda}{y^2} \quad (2)$

$\frac{2}{x} + \frac{4}{y} = 3 \qquad \frac{2}{x} + \frac{4}{y} = 3 \quad (3)$

(1) i (2) $\Rightarrow \frac{\lambda}{x^2} = \frac{\lambda}{y^2} \Rightarrow x^2 = y^2$

tj. $x = \pm y$

za $x = y$ iz (3) $\quad \frac{2}{x} + \frac{4}{x} = 3$

$\frac{6}{x} = 3 \Rightarrow x = 2 \Rightarrow y = 2$

za $x = -y$ iz (3)

$\frac{2}{x} - \frac{4}{x} = 3 \Rightarrow -\frac{2}{x} = 3$

$3x = -2 \Rightarrow x = -\frac{2}{3}$

$\Rightarrow y = \frac{2}{3}$

za $M_1(2,2) \Rightarrow 2 - 2\lambda \cdot \frac{1}{4} = 0$

$\lambda = 4$

za $M_2\left(-\frac{2}{3}, \frac{2}{3}\right) \Rightarrow 2 - 2\lambda \cdot \frac{9}{4} = 0 \Rightarrow \lambda = \frac{4}{9}$

Stacionarne tačke su $M_1(2,2)$ za $\lambda = 4$ i $M_2\left(-\frac{2}{3}, \frac{2}{3}\right)$ za $\lambda = \frac{4}{9}$.

$\frac{\partial^2 F}{\partial x^2} = \frac{4\lambda}{x^3}$

Za $M_1(2,2)$, $\lambda = 4$

$A = \frac{16}{8} = 2$, $B = 0$, $C = \frac{32}{8} = 4$, $D = AC - B^2 = 8 > 0$ f-ja ima ekstrem

$\frac{\partial^2 F}{\partial x \partial y} = 0$

$A > 0 \Rightarrow$ f-ja ima minimum

$\frac{\partial^2 F}{\partial y^2} = \frac{8\lambda}{y^3}$

$z_{min}(2,2) = 4 + 8 = 12$

Za $M_2\left(-\frac{2}{3}, \frac{2}{3}\right)$, $\lambda = \frac{4}{9}$, $A = \frac{\frac{16}{9}}{-\frac{8}{27}} = -\frac{16 \cdot 27}{8 \cdot 9} = -2 \cdot 3 = -6$

$B = 0$, $C = \frac{\frac{32}{9}}{\frac{8}{27}} = \frac{32 \cdot 27}{8 \cdot 9} = 4 \cdot 3 = 12$, $D = AC - B^2 = -72 < 0 \Rightarrow$

$\Rightarrow$ f-ja u tački M_2 nema ekstremnu vrijednost

(#) Naći uslovne ekstreme f-je $z=xy$ ako je

$x^2+y^2=2ax$, $a>0$.

Rj: Posmatramo f-ju $F(x,y,\lambda)=xy+\lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x}=y+2\lambda x-2a\lambda$$

$$\frac{\partial F}{\partial y}=x+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-2ax$$

$$y+2\lambda x-2a\lambda=0$$
$$x+2\lambda y=0$$
$$x^2+y^2-2a\cdot x=0$$

(1) $y+2\lambda(x-a)=0 \Rightarrow x-a=\frac{-y}{2\lambda}$ ...(*)

(2) $x=-2\lambda y$

(3) $x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

(2) u (1): $y+2\lambda(-2\lambda y-a)=0$

(3): $(x-a)^2+y^2=a^2$

$$y-4\lambda^2y-2a\lambda=0$$
$$y(1-4\lambda^2)=2a\lambda$$
$$y=\frac{2a\lambda}{1-4\lambda^2}$$

(*) u (3):

$$\frac{y^2}{4\lambda^2}+y^2=a^2$$
$$y^2\left(\frac{1}{4\lambda^2}+1\right)=a^2$$
$$y^2\left(\frac{1+4\lambda^2}{4\lambda^2}\right)=a^2$$
$$y^2=\frac{4a^2\lambda^2}{1+4\lambda^2}$$
$$y=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}}$$

$$y=\frac{2a\lambda}{1-4\lambda^2}=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1-4\lambda^2=\pm\sqrt{1+4\lambda^2}$$

$$(1-4\lambda^2)^2=1+4\lambda^2$$
$$16\lambda^4-8\lambda^2+1=1+4\lambda^2$$
$$16\lambda^4-12\lambda^2=0$$
$$\lambda^2(16\lambda^2-12)=0$$

$\lambda_1=0$

$\lambda_{2,3}=\pm\sqrt{\frac{12}{16}}=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$

$\lambda_1=0$: $y=0$, $x=0$

$\lambda_2=\frac{\sqrt{3}}{2}$:

$$y+\sqrt{3}x-a\sqrt{3}=0$$
$$x+y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$\sqrt{3}x+y=a\sqrt{3}$$
$$-\ \sqrt{3}x+3y=0$$
$$-2y=a\sqrt{3}$$
$$y=\frac{-a}{2}\sqrt{3} \qquad x=\frac{3}{2}a$$

$\lambda_3=-\frac{\sqrt{3}}{2}$:

$$y-x\sqrt{3}+a\sqrt{3}=0$$
$$x-y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$-x\sqrt{3}+y=-a\sqrt{3}$$
$$+\ x\sqrt{3}-3y=0$$
$$-2y=-a\sqrt{3}$$
$$y=\frac{\sqrt{3}}{2}a \Rightarrow x=\frac{3}{2}a$$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$ za $\lambda=\frac{\sqrt{3}}{2}$ i $M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$.

$$\frac{\partial^2 F}{\partial x^2}=2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y}=1$$

$$\frac{\partial^2 F}{\partial y^2}=2\lambda$$

$M_1(0,0)$, $\lambda=0$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$, $\lambda=\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A=\sqrt{3}>0 \Rightarrow$ f-ja ima minimum

$$Z_{min}\left(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a\right)=-\frac{3\sqrt{3}}{4}a^2$$

$M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt{3}<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

$$Z_{max}\left(\frac{3}{2}a, \frac{\sqrt{3}}{2}a\right)=\frac{3\sqrt{3}}{4}a^2$$